Conforming lifting and adjoint consistency for the Discrete de Rham complex of differential forms

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1 The Discrete de Rham construction

2 Objective and motivation

3 Construction of the Lifting

4 Solving the two abstract problems

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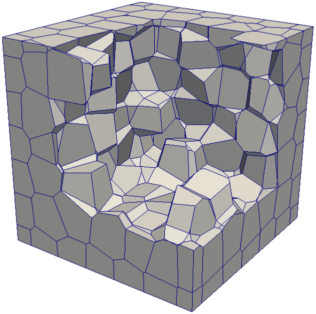
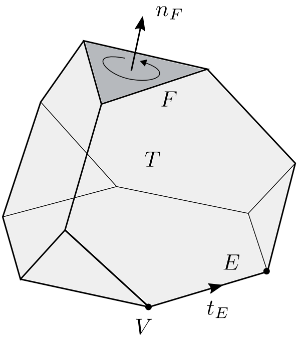
Domain and polytopal mesh

- $\Omega \subset \mathbb{R}^n$ polyhedral domain
- $\mathcal{M}_h$ polyhedral mesh on Ω
- $\Delta_d(\mathcal{M}_h)$ set of d-cells





Discrete de Rham method with differential forms

- The de Rham complex with differential forms:

$$H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}^0} \cdots \xrightarrow{\mathrm{d}^{k-1}} H\Lambda^k(\Omega) \xrightarrow{\mathrm{d}^k} \cdots \xrightarrow{\mathrm{d}^{n-1}} H\Lambda^n(\Omega)$$

- The DDR complex of **polynomial differential forms** ([1]):

$$\underline{X}^0_{r,h} \xrightarrow{\underline{\mathrm{d}}^0_{r,h}} \cdots \xrightarrow{\underline{\mathrm{d}}^{k-1}_{r,h}} \underline{X}^k_{r,h} \xrightarrow{\underline{\mathrm{d}}^k_{r,h}} \cdots \xrightarrow{\underline{\mathrm{d}}^{n-1}_{r,h}} \underline{X}^n_{r,h}$$

Design principle: recursive construction on d-cells (e.g., k-form on $d = k+1, \ldots, n$) mimicking the Stokes formula

[1] [Bonaldi et al., 2023]

Trimmed polynomial spaces

- Let $f \in \Delta_d(\mathcal{M}_h)$, fix $\boldsymbol{x}_f \in f$
- **Koszul complement**:

$$\mathcal{K}_r^\ell(f) := \kappa \mathcal{P}_{r-1}\Lambda^{\ell+1}(f)$$

with $(\kappa\omega)_{\boldsymbol{x}}(\cdot, \ldots) := \omega_{\boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_f, \ldots)$

Trimmed polynomial spaces

- Let $f \in \Delta_d(\mathcal{M}_h)$, fix $\boldsymbol{x}_f \in f$
- **Koszul complement**:

$$\mathcal{K}_r^\ell(f) \coloneqq \kappa \mathcal{P}_{r-1}\Lambda^{\ell+1}(f)$$

with $(\kappa\omega)_{\boldsymbol{x}}(\cdot, \ldots) \coloneqq \omega_{\boldsymbol{x}}(\boldsymbol{x} - \boldsymbol{x}_f, \ldots)$

- **Trimmed polynomial spaces** $(\ell \geq 1)$:

$$\begin{aligned} \mathcal{P}_r^-\Lambda^0(f) &\coloneqq \mathcal{P}_r\Lambda^0(f), \\ \mathcal{P}_r^-\Lambda^\ell(f) &\coloneqq \mathrm{d}\mathcal{P}_r\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f) \end{aligned}$$

Remarks:

- *Full space:* $\mathcal{P}_r\Lambda^\ell(f) = \mathrm{d}\mathcal{P}_{r+1}\Lambda^{\ell-1}(f) \oplus \mathcal{K}_r^\ell(f)$

Discrete local potential and exterior derivative (I)

- Discrete potential (playing the role of a k-form inside f)

$$P^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^k(f)$$

- Discrete exterior derivative

$$\mathrm{d}^k_{r,f} : \underline{X}^k_{r,f} \to \mathcal{P}_r\Lambda^{k+1}(f)$$

Discrete local potential and exterior derivative (I)

For $d = k, \ldots, n$, all $f \in \Delta_d(\mathcal{M}_h)$, and $\underline{\omega}_f \in \underline{X}^k_{r,f}$:

- **Case** $d = k$:

$$P^k_{r,f}\underline{\omega}_f := \star^{-1}\omega_f \in \mathcal{P}_r\Lambda^d(f)$$

- **Case** $d \geq k+1$:
  - Discrete exterior derivative $\mathrm{d}^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^{k+1}(f)$ defined such that, $\forall \mu \in \mathcal{P}_r\Lambda^{d-k-1}(f)$:

$$\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu = (-1)^{k+1}\int_f \star^{-1}\omega_f \wedge \mathrm{d}\mu + \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

Discrete local potential and exterior derivative (II)

Case $d \geq k+1$ (continued):

- Discrete potential $P^k_{r,f}\underline{\omega}_f \in \mathcal{P}_r\Lambda^k(f)$ defined using the decomposition $\mathcal{P}_r\Lambda^{d-k}(f) = \mathrm{d}\mathcal{K}^{d-k-1}_{r+1}(f) \oplus \mathcal{K}^{d-k}_r(f)$:
  - For all $\mu \in \mathcal{K}^{d-k-1}_{r+1}(f)$:

$$(-1)^{k+1}\int_f P^k_{r,f}\underline{\omega}_f \wedge \mathrm{d}\mu = \int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu - \int_{\partial f} P^k_{r,\partial f}\underline{\omega}_{\partial f} \wedge \mathrm{tr}_{\partial f}\,\mu$$

  - For all $\nu \in \mathcal{K}^{d-k}_r(f)$:

$$\int_f \star^{-1} P^k_{r,f}\underline{\omega}_f \wedge \nu = \int_f \star^{-1}\omega_f \wedge \nu$$

Discrete local potential and exterior derivative (III)

Lemma (Polynomial consistency)

For all integers $0 \leq k \leq d \leq n$ *and all* $f \in \Delta_d(\mathcal{M}_h)$*, it holds*

$$P^k_{r,f}\underline{I}^k_{r,f}\omega = \omega \qquad \forall \omega \in \mathcal{P}_r\Lambda^k(f),$$

and, if $d \geq k+1$*,*

$$\mathrm{d}^k_{r,f}\underline{I}^k_{r,f}\omega = \mathrm{d}\omega \qquad \forall \omega \in \mathcal{P}^-_{r+1}\Lambda^k(f).$$

Discrete $X_{r,h}^k$ spaces : hierarchical structure

$$\underline{X}_{r,h}^k := \bigtimes_{d=k}^{n} \bigtimes_{f\in\Delta_d(\mathcal{M}_h)} \star^{-1}\mathcal{P}_r^-\Lambda^{d-k}(f)$$

For $\omega \in \Lambda^k(\Omega)$, $\quad \underline{I}_{r,h}^k\omega = \left(\pi_{r,f}^{-,d-k}(\star\,\mathrm{tr}_f\,\omega)\right)_{f\in\Delta_k(\mathcal{M}_h)} \in \underline{X}_{r,h}^k$

Space	$f_0 \equiv V$	$f_1 \equiv E$	$f_2 \equiv F$	$f_3 \equiv T$
$\underline{X}_{r,h}^0$	$\mathbb{R} = \star^{-1}\mathcal{P}_r\Lambda^0(f_0)$	$\star^{-1}\mathcal{P}_r^-\Lambda^1(f_1)$	$\star^{-1}\mathcal{P}_r^-\Lambda^2(f_2)$	$\dots$
$\underline{X}_{r,h}^1$		$\star^{-1}\mathcal{P}_r\Lambda^0(f_1)$	$\star^{-1}\mathcal{P}_r^-\Lambda^1(f_2)$	$\dots$
$\underline{X}_{r,h}^2$			$\star^{-1}\mathcal{P}_r\Lambda^0(f_2)$	$\dots$
$\underline{X}_{r,h}^3$				$\dots$

Discrete $X^k_{r,h}$ spaces : L^2-products

- We can define on $\underline{X}^k_{r,h}$ a discrete L^2-product:

$$(\underline{\omega}_h, \underline{\mu}_h)_{k,h} \coloneqq \sum_{f \in \Delta_n(\mathcal{M}_h)} \left(\int_f P^k_{r,f} \underline{\omega}_f \wedge \star P^k_{r,f} \underline{\mu}_f + s_{k,f}(\underline{\omega}_f, \underline{\mu}_f) \right)$$

- Above, $s_{k,f}$ is a **stabilisation term** such that

$$s_{k,f}(\underline{I}^k_{r,f}\omega, \underline{\mu}_f) = 0 \qquad \forall \omega \in \mathcal{P}_r \Lambda^k(f)$$

Global discrete exterior derivative and DDR complex

- Global discrete exterior derivative: $\underline{\mathrm{d}}_{r,h}^{k} : \underline{X}_{r,h}^{k} \to \underline{X}_{r,h}^{k+1}$

$$\underline{\mathrm{d}}_{r,h}^{k}\underline{\omega}_h := \left(\pi_{r,f}^{-,d-k-1}(\star \mathrm{d}_{r,f}^{k}\underline{\omega}_f)\right)_{f \in \Delta_{[k+1\dots n]}(\mathcal{M}_h)}$$

- DDR sequence:

$$\underline{X}_{r,h}^{0} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{0}} \cdots \xrightarrow{\underline{\mathrm{d}}_{r,h}^{k-1}} \underline{X}_{r,h}^{k} \xrightarrow{\underline{\mathrm{d}}_{r,h}^{k}} \cdots \xrightarrow{\underline{\mathrm{d}}_{r,h}^{n-1}} \underline{X}_{r,h}^{n}$$

Global discrete exterior derivative and DDR complex

- DDR sequence:

$$\underline{X}_{r,h}^0 \xrightarrow{\underline{\mathrm{d}}_{r,h}^0} \cdots \xrightarrow{\underline{\mathrm{d}}_{r,h}^{k-1}} \underline{X}_{r,h}^k \xrightarrow{\underline{\mathrm{d}}_{r,h}^k} \cdots \xrightarrow{\underline{\mathrm{d}}_{r,h}^{n-1}} \underline{X}_{r,h}^n$$

Lemma (Cohomology of the Discrete de Rham complex)

The DDR sequence is a complex and its cohomology is isomorphic to the cohomology of the continuous de Rham complex, i.e., for all k,

$$\mathrm{Ker}\,\underline{\mathrm{d}}_{r,h}^k / \mathrm{Im}\,\underline{\mathrm{d}}_{r,h}^{k-1} \cong \mathrm{Ker}\,\mathrm{d}^k / \mathrm{Im}\,\mathrm{d}^{k-1}.$$

Global discrete exterior derivative and DDR complex

- DDR sequence:

$$\underline{X}^0_{r,h} \xrightarrow{\underline{\mathrm{d}}^0_{r,h}} \cdots \xrightarrow{\underline{\mathrm{d}}^{k-1}_{r,h}} \underline{X}^k_{r,h} \xrightarrow{\underline{\mathrm{d}}^k_{r,h}} \cdots \xrightarrow{\underline{\mathrm{d}}^{n-1}_{r,h}} \underline{X}^n_{r,h}$$

Lemma (Poincaré inequalities)

For any $k = 0, \ldots, n-1$ and $\underline{\omega}_h \in \underline{X}^k_{r,h}$, there exists $\underline{\tau}_h \in \underline{X}^k_{r,h}$ such that $\underline{\mathrm{d}}^k_{r,h}\underline{\tau}_h = \underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h$ and

$$|||\underline{\tau}_h|||_h \lesssim |||\underline{\mathrm{d}}^k_{r,h}\underline{\omega}_h|||_h.$$

Objective and motivation

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Goal: conforming lifting operator

Build a linear operator

$$\mathcal{L}^k_{r,h} : \underline{X}^k_{r,h} \to H\Lambda^k(\Omega)$$

Key Properties: for each $d \in \{k, \ldots, n\}$ and $f \in \Delta_d(\mathcal{M}_h)$,

- **Projection**: for all $\zeta \in \mathcal{P}^-_{r+1}\Lambda^{d-k}(f)$,

$$\int_f \operatorname{tr}_f(\mathcal{L}^k_{r,h}\underline{\omega}_h) \wedge \zeta = \int_f P^k_{r,f}\underline{\omega}_f \wedge \zeta$$

- **Stability**:

$$\|\operatorname{tr}_f(\mathcal{L}^k_{r,h}\underline{\omega}_h)\|_f \lesssim |||\underline{\omega}_f|||_f$$

$$\|\mathrm{d}\operatorname{tr}_f(\mathcal{L}^k_{r,h}\underline{\omega}_h)\|_f \lesssim |||\underline{\mathrm{d}}^k_{r,f}\underline{\omega}_f|||_f$$

Motivation : proof of adjoint consistency (I)

Adjoint consistency: $\alpha \in H\Lambda^k(\Omega)$ continuous k-form, $\underline{\omega}_h \in \underline{X}_{r,h}^{n-k-1}$,

$$\underbrace{\left| \int_\Omega \alpha \wedge \underline{\mathrm{d}}_{r,h}^{n-k-1} \underline{\omega}_h - (-1)^{k+1} \int_\Omega \mathrm{d}\alpha \wedge P_{r,h}^{n-k-1} \underline{\omega}_h \right|}_{\mathfrak{I}} \lesssim h^{r+1} \mathcal{N}(\alpha, \underline{\omega}_h)$$

Motivation : proof of adjoint consistency (II)

Proof steps:

1. Introduce local polynomial approximations $(\mu_f)_{f\in\mathcal{T}_h}$ of α:

$$\mathfrak{I} = \sum_{f\in\Delta_n(\mathcal{M}_h)} \Bigg(\int_f (\alpha - \mu_f) \wedge \underline{\mathrm{d}}_{r,f}^{n-k-1}\underline{\omega}_f + (-1)^{k+1} \int_f \mathrm{d}(\mu_f - \alpha) \wedge P_{r,f}^{n-k-1}\underline{\omega}_f + (-1)^k \sum_{f'\in\Delta_{n-1}(f)} \epsilon_{ff'} \int_{f'} \mathrm{tr}_{f'}(\mu_f) \wedge P_{r,f'}^{n-k-1}\underline{\omega}_{f'} \Bigg)$$

2. Introduce $\mathcal{L}_{r,h}^{n-k-1}$, replace potential $P_{,}$ with lifting trace:

$$P_{r,f'}^{n-k-1}\underline{\omega}_{f'} \longrightarrow \mathrm{tr}_{f'}(\mathcal{L}_{r,h}^{n-k-1}\underline{\omega}_h)$$

3. Apply **Integration By Parts** on the continuous lifting to close the estimate

Motivation : stabilization free DDR methods

- Principle: the bilinear form is built by evaluating the continuous operator directly on the lifting $\mathcal{L}^k_{r,h}$
- No stabilization term: coercivity is assured by the piecewise definition and properties of $\mathcal{L}^k_{r,h}$

Example: the discrete bilinear form for the Hodge-Laplace problem reads

$$a_h(\underline{\omega}_h, \underline{\nu}_h) := \int_\Omega \mathrm{d}\left(\mathcal{L}^k_{r,h}\underline{\omega}_h\right) \wedge \star\mathrm{d}\left(\mathcal{L}^k_{r,h}\underline{\nu}_h\right)$$

Construction of the Lifting

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Principle behind the construction

Recursive construction: on cell dimension d, from k to n

- **Base case** $d = k$:

$$\text{Define } \mathcal{L}^k_{r,f}\underline{\omega}_f := P^k_{r,f}\underline{\omega}_f$$

- **Inductive step** $d > k$:
  - Find $\lambda \in H\Lambda^k(f)$ such that
$$\begin{cases} \mathrm{d}\lambda = \mathrm{d}^k_{r,f}\underline{\omega}_f + \mathcal{C}^k_{r,f}\underline{\omega}_f \\ \mathrm{tr}_{\partial f}\,\lambda = \mathcal{L}^k_{r,\partial f}\underline{\omega}_{\partial f} \end{cases}$$
    with $\mathcal{C}^k_{r,f}$ a **lifting correction** (non-zero only if $d \geq k+2$)
  - Find $\tau \in H\Lambda^{k-1}_0(f)$ such that
$$\int_f \mathrm{d}\tau \wedge \nu = \int_f \left(P^k_{r,f}\underline{\omega}_f - \lambda\right) \wedge \nu, \quad \forall \nu \in \mathcal{K}^{d-k}_r(f)$$

$$\text{Define } \mathcal{L}^k_{r,f}\underline{\omega}_f := \lambda + \mathrm{d}\tau$$

Verification of properties: Projection

Goal: Prove the **projection property** for all $\zeta \in \mathcal{P}^{-}_{r+1}\Lambda^{d-k}(f)$

$$\int_f \mathrm{tr}_f(\mathcal{L}^k_{r,h}\underline{\omega}_h) \wedge \zeta = \int_f P^k_{r,f}\underline{\omega}_f \wedge \zeta$$

Proof sketch: Use decomposition $\zeta = \mathrm{d}\mu + \nu$

- **Step 1**: For $\mathrm{d}\mu$ (with $\mu \in \mathcal{K}^{d-k}_r(f)$):

$$\begin{aligned}\int_f \mathcal{L}^k_{r,f}\underline{\omega}_f \wedge \mathrm{d}\mu &= (-1)^{k+1}\int_f \mathrm{d}\mathcal{L}^k_{r,f}\underline{\omega}_f \wedge \mu + \text{boundary terms} \\ &\overset{(*)}{=} (-1)^{k+1}\int_f \mathrm{d}^k_{r,f}\underline{\omega}_f \wedge \mu + \dots \\ &= \int_f P^k_{r,f}\underline{\omega}_f \wedge \mathrm{d}\mu\end{aligned}$$

(*) using $\mathrm{d}\mathcal{L}^k_{r,f} = \mathrm{d}^k_{r,f} + \mathcal{C}^k_{r,f}$

Verification of properties: Projection

Goal: Prove the **projection property** for all $\zeta \in \mathcal{P}^-_{r+1}\Lambda^{d-k}(f)$

$$\int_f \mathrm{tr}_f(\mathcal{L}^k_{r,h}\underline{\omega}_h) \wedge \zeta = \int_f P^k_{r,f}\underline{\omega}_f \wedge \zeta$$

Proof sketch: Use decomposition $\zeta = \mathrm{d}\mu + \nu$

- **Step 2**: For $\nu \in \mathcal{K}^{d-k+1}_r(f)$:

$$\int_f (\lambda + \mathrm{d}\tau) \wedge \nu = \int_f P^k_{r,f}\underline{\omega}_f \wedge \nu$$

  Guaranteed by the construction of τ.

Verification of properties: Stability

Goal: Prove the bound $\| \operatorname{tr}_f(\mathcal{L}^k_{r,h}\underline{\omega}_h)\|_f \lesssim |||\underline{\omega}_f|||_f$

Proof idea: The lifting is defined as $\mathcal{L}^k_{r,f}\underline{\omega}_f = \lambda + \mathrm{d}\tau$

1. λ is the **minimal norm solution** to a potential problem $\Rightarrow$ its norm is controlled by the data (RHS and boundary conditions)

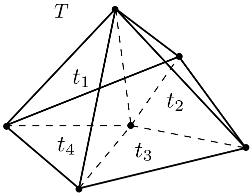
$$\|\lambda\|_f \lesssim \|\mathrm{d}^k_{r,f}\underline{\omega}_f\|_f + \|\mathcal{L}^k_{r,\partial f}\underline{\omega}_{\partial f}\|_{\partial f}$$

2. $\mathrm{d}\tau$ has zero trace on ∂f and is constructed to match the projection $\Rightarrow$

$$\|\mathrm{d}\tau\|_f \lesssim \|P^k_{r,f}\underline{\omega}_f - \lambda\|_f$$

A preliminary step: FEEC spaces on a simplicial submesh

Simplicial submesh: simplicial subdivision $\mathcal{S}_h(f)$ of each cell $f \in \Delta_d(\mathcal{M}_h)$



FEEC spaces on $\mathcal{S}_h(f)$:

$$\dots \longrightarrow \mathcal{P}_r^- \Lambda^k(S_h(f)) \xrightarrow{\mathrm{d}} \mathcal{P}_r^- \Lambda^{k+1}(S_h(f)) \longrightarrow \dots$$

Towards two abstract problems

Let $f \in \Delta_d(\mathcal{M}_h)$

Data: $\xi \in \mathcal{P}_r^- \Lambda^{k+1}(\mathcal{S}_h(f))$, $\theta \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{S}_h(\partial f))$, $\zeta \in L^2\Lambda^k(f)$

Find:

1. $\lambda \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{S}_h(f))$ such that

$$\begin{cases} \mathrm{d}\lambda = \xi \\ \mathrm{tr}_{\partial f}\, \lambda = \theta \end{cases}$$

2. $\tau \in \mathring{\mathcal{P}}_?^- \Lambda^{k-1}(\mathcal{S}_h(f))$ such that, for all d-simplex $t \in \mathcal{S}_h(f)$,

$$-\int_t \tau \wedge \mathrm{d}z = \int_t \zeta \wedge z, \quad \forall z \in \mathcal{K}_r^{d-k}(t)$$

Towards two abstract problems

Let $f \in \Delta_d(\mathcal{M}_h)$

Data: $\xi \in \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{S}_h(f))$, $\theta \in \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{S}_h(\partial f))$, $\zeta \in L^2\Lambda^k(f)$

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$$-\int_t \tau \wedge \mathrm{d}z = \int_t \zeta \wedge z, \quad \forall z \in \mathcal{K}_r^{d-k}(t)$$

Novel aspects of the construction

- Simplicial submesh + Dropping co-differential constraint (i.e., not solving **curl**-div systems) $\Rightarrow$ no need to assume solution regularity

- Explicit polynomial expression $\Rightarrow$ convenient to work with and allows for straightforward stability estimates

Solving the two abstract problems

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Reminder : two abstract problems

Let $f \in \Delta_d(\mathcal{M}_h)$,

Data: $\xi \in \mathcal{P}_r^- \Lambda^{k+1}(\mathcal{S}_h(f))$, $\theta \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{S}_h(\partial f))$, $\zeta \in L^2\Lambda^k(f)$

Find:

1. $\lambda \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{S}_h(f))$ such that

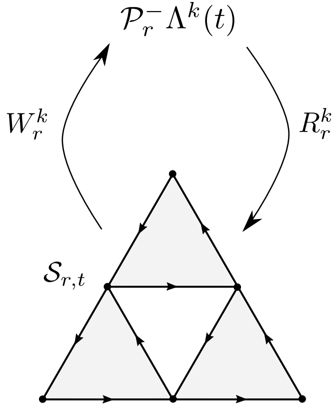
$$\begin{cases} \mathrm{d}\lambda = \xi \\ \mathrm{tr}_{\partial f}\, \lambda = \theta \end{cases}$$

2. $\tau \in \mathring{\mathcal{P}}_?^- \Lambda^{k-1}(\mathcal{S}_h(f))$ such that, for all d-simplex $t \in \mathcal{S}_h(f)$,

$$-\int_t \tau \wedge \mathrm{d}z = \int_t \zeta \wedge z, \quad \forall z \in \mathcal{K}_k^\ell(t)$$

First problem: solving via small simplices

Let $t \in \Delta_d(\mathcal{S}_h(f))$



Cochain spaces: $C^k(\mathcal{S}_{r,t}) \cong \mathbb{R}^{\mathrm{card}(\Delta_k(\mathcal{S}_{r,t}))}$

DoFs on small simplices

DoFs on small simplices

Define

$$C^k(\mathcal{S}_{r,t}, \mathcal{H}_{r,t}) := \operatorname{Im} R_r^k.$$

Then the graded spaces $C^k(\mathcal{S}_{r,t}, \mathcal{H}_{r,t})$, together with the differential δ^k (**coboundary operator**), form a cochain subcomplex of $C^k(\mathcal{S}_{r,t})$. Moreover, there exists a **cochain isomorphism**

$$\varphi_r^k : C^k(\mathcal{S}_{r,t}, \mathcal{H}_{r,t}) \to \mathcal{P}_r^- \Lambda^k(t)$$

An equivalent problem on small simplices

For $f \in \Delta_d(\mathcal{M}_h)$

Data: $\tilde{\xi} \in C^{k+1}(\mathcal{S}_{r,f}, \mathcal{H}_{r,f})$ and $\tilde{\theta} \in C^k(\mathcal{S}_{r,f}, \mathcal{H}_{r,f})$

Find: $\tilde{\lambda} \in C^k(\mathcal{S}_{r,f}, \mathcal{H}_{r,f})$ such that

$$\begin{cases} \delta^k \tilde{\lambda} = \tilde{\xi}, \\ \gamma^k_{\partial f} \tilde{\lambda} = \tilde{\theta} \end{cases}$$

Existence is proven by an explicit construction of a solution !

Let $f \in \Delta_d(\mathcal{M}_h)$

Data: $\xi \in \mathcal{P}_r^- \Lambda^{k+1}(\mathcal{S}_h(f))$, $\theta \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{S}_h(\partial f))$, $\zeta \in L^2\Lambda^k(f)$

Find:

1. $\lambda \in \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{S}_h(f))$ such that

$$\begin{cases} \mathrm{d}\lambda = \xi \\ \mathrm{tr}_{\partial f} \lambda = \theta \end{cases}$$

2. $\tau \in \mathring{\mathcal{P}}_?^- \Lambda^{k-1}(\mathcal{S}_h(f))$ such that, for all d-simplex $t \in \mathcal{S}_h(f)$,

$$-\int_t \tau \wedge \mathrm{d}z = \int_t \zeta \wedge z, \quad \forall z \in \mathcal{K}_r^{d-k}(t)$$

Second problem

Hodge duality on FEEC spaces

Let t be a d-simplex. Then, for any nonzero $\alpha \in \mathcal{P}_r\Lambda^k(t)$ there exists a form $\beta \in \mathring{\mathcal{P}}^-_{r+d-k+1}\Lambda^{d-k}(t)$ such that

$$\int_t \beta \wedge \alpha > 0$$

Corollary: the bilinear form $\int_t(\cdot,\cdot)$ is non-degenerate, and for the linear functional

$$L(\mathrm{d}z) = \int_t \zeta \wedge z, \qquad z \in \mathcal{K}^{d-k}_r(t)$$

there exists a unique $\tau \in \mathring{\mathcal{P}}^-_{r+d-k+1}\Lambda^{k-1}(t)$ such that

$$\int_t \tau \wedge \mathrm{d}z = L(\mathrm{d}z).$$

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Take-home message

- General strategy: Solving underdetermined potential problems via FEEC spaces on simplicial submeshes

- Corollaries:
  - A framework to establish analytical results on polytopal complexes (e.g., Poincaré inequalities, adjoint consistency, compactness)
  - Explicit and computable conforming representations of discrete data (enabling stabilization-free FEM on polyhedral meshes)

Thank you for your attention!

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Selected Publications

Di Pietro, D. A., Droniou, J., & Pitassi, S. (2025).

Conforming lifting and adjoint consistency for the Discrete de Rham complex of differential forms.

arXiv preprint

Alonso Rodríguez, A., Pitassi, S., & Rapetti, F.

Poincaré-constants estimates on Finite Element simplicial stars: construction via high-order potentials.

In preparation

References I

Bonaldi, F., Di Pietro, D. A., Droniou, J., and Hu, K. (2023).
An exterior calculus framework for polytopal methods.
http://arxiv.org/abs/2303.11093.