Generators of $H^1(\Gamma, \partial\Gamma^{\mathrm{c}})$ with $\partial\Gamma^{\mathrm{c}} \subset \partial\Gamma$ for Triangulated Surfaces Γ: Construction and Classification of Global Loops

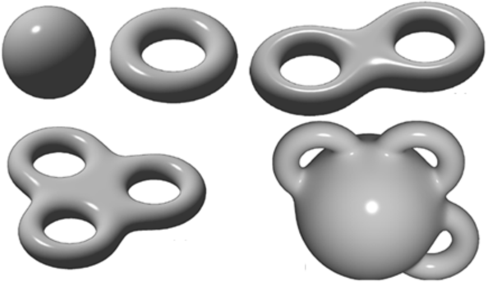
Silvano Pitassi

IMAG, University of Montpellier, France

Introduction

Motivation: Electromagnetic scattering

Let Γ be a triangulated embedded surface (i.e. a closed 2-manifold)



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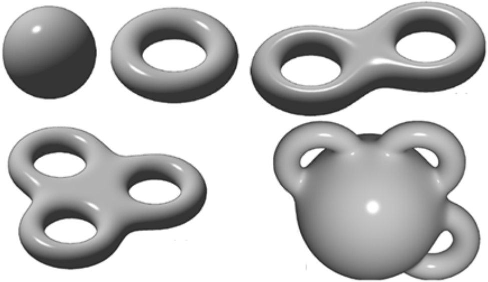


- **Electrical Field Integral Equation (EFIE)**: find the induced current $\boldsymbol{J}$ on Γ from an *incident* (source) electromagnetic field $\boldsymbol{E}^i$

$$\gamma_{\mathrm{T}} \circ \mathfrak{U}_k(\boldsymbol{J}) = -\gamma_{\mathrm{T}} \boldsymbol{E}^i$$

Motivation: Low-frequency breakdown

- **Problem**: poor conditioning of BEM matrices and numerical instability as $\omega \to 0$

Motivation: Low-frequency breakdown

- **Problem**: poor conditioning of BEM matrices and numerical instability as $\omega \to 0$

- **Loop-Star decomposition**: decompose the current $\boldsymbol{J}$ as

$$\boldsymbol{J} = \underbrace{\boldsymbol{J}_{\text{loop}}}_{\text{Solenoidal}} + \underbrace{\boldsymbol{J}_{\text{star}}}_{\text{Non-solenoidal}}$$

and introduce the frequency-dependent scalings

$$\boldsymbol{J}_{\text{loop}} = \tilde{\boldsymbol{J}}_{\text{loop}} \quad \text{and} \quad \boldsymbol{J}_{\text{star}} = \omega \tilde{\boldsymbol{J}}_{\text{star}}$$

Why Cohomology?

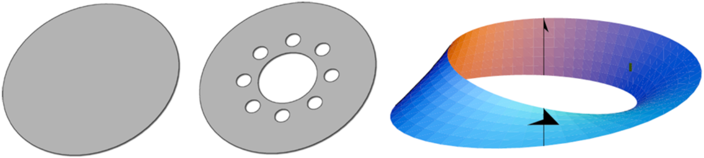
$\boldsymbol{J}_{\text{loop}}$ and $\boldsymbol{J}_{\text{star}}$ are determined as

$$\boldsymbol{J} = \underbrace{\operatorname{curl}_\Gamma \psi + \boldsymbol{g}}_{=:\boldsymbol{J}_{\text{loop}}} + \underbrace{\operatorname{grad}_\Gamma \varphi}_{=:\boldsymbol{J}_{\text{Star}}},$$

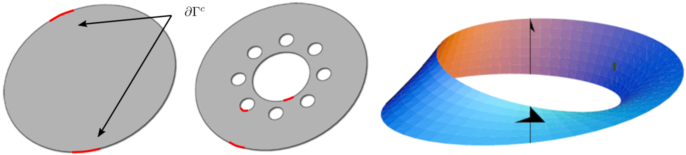
with $\boldsymbol{g} \in H^1(\Gamma)$ a **cohomology generator** (**global loop**)

The general problem

- Surface with boundary



- Presence of "contacts" $\partial\Gamma^c$



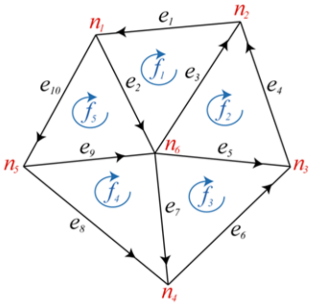
We need a general algorithm to compute generators of the **first relative cohomology group** $H^1(\Gamma, \partial\Gamma^c)$

Key Mathematical Concepts

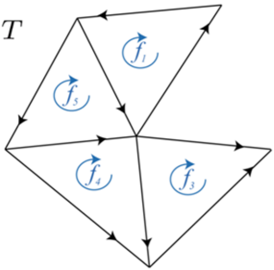
Chains

Formal **linear combinations** of k-cells with coefficients in an commutative ring G ($\mathbb{Z}$ or $\mathbb{R}$)

Example:



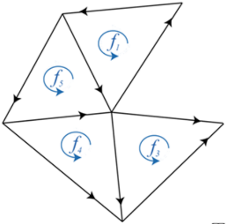
$$w = \{1, 0, 1, 1, 1\}^T$$



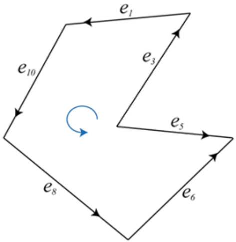
Chains can be added and the k-cells are a basis for the **chain group** $C_k(\Gamma, G)$

Boundary operator

Boundary operator: $\partial_k : C_k(\Gamma; \mathbb{Z}) \to C_{k-1}(\Gamma; \mathbb{Z})$



$$W = \{1, 0, 1, 1, 1\}^T$$



$$\partial_2 = \begin{array}{cc} \textbf{Number of faces} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \textbf{Number of edges} \end{array}$$

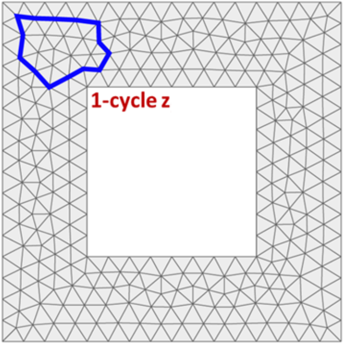
$$Z = \partial_2 W = \{1, 0, 1, 0, -1, 1, 0, 1, 0, 1\}^T$$

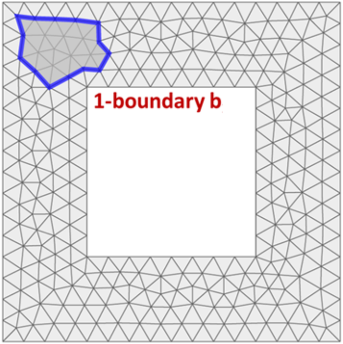
Cycles and boundaries

Boundary operator ∂_k gives rise to a classification of k-chains

Example: $k = 1$

- 1-chains z whose **boundary is zero** $\partial_1 z = 0$ are 1-cycles in $Z_1(\Gamma)$
- 1-chains b **that are boundary** of a 2-chain $\partial_2 c = b$ are 1-boundaries in $B_1(\Gamma)$

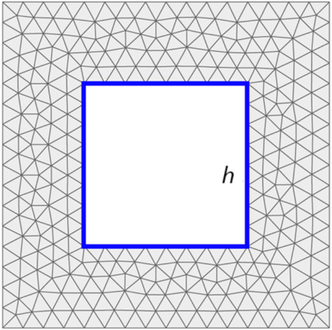




First homology group

All 1-boundaries are 1-cycles, but the converse is not true

Example: hollow disk



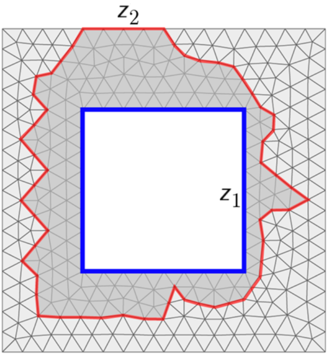
The first homology group is defined as the **quotient group**

$$H_1(\Gamma) := Z_1(\Gamma)/B_1(\Gamma)$$

Homology class

Equivalence relation: two 1-cycles z_1 and z_2 are homologous if their difference is a 1-boundary: $z_1 \sim z_2 \iff z_1 - z_2 \in B_1(\Gamma)$

Example:



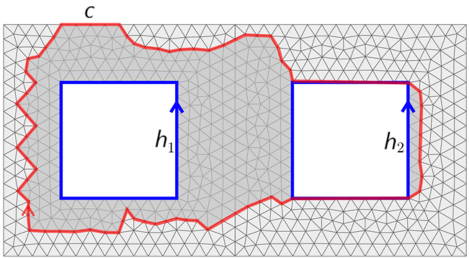
z_1 and z_2 are in the same **homology class** $[z_1]$. Each homology class is represented by any of its elements (called **representative**)

Homology generators and homology basis

Homology generators: a minimal set of homology classes $[h_i]$ such that

$$c = \sum_{i=1}^{\beta_1(\Gamma)} a_i h_i + b$$

Example:



Cochains

A k-cochain $\mathbf{c}$ is a **linear functional** $\mathbf{c} : C_k(\Gamma; G) \to G$

Cochains can be added in the natural way and form the **cochain group** $C^k(\Gamma; G)$ (**dual space** of $C_k(\Gamma; G)$)

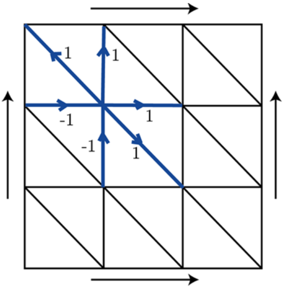
Coboundary, cocycles and coboundaries

The coboundary operator δ^k $C^k(\Gamma; G) \to C^{k+1}(\Gamma; G)$ is designed in such a way that the **Generalized Stokes theorem** holds:

$$< \delta^k \mathbf{c}, w > = < \mathbf{c}, \partial_{k+1} w >$$

δ^k gives rise to a classification of 1-cochains:

1. 1-cochains **c** whose **coboundary is zero** $\delta^1 \mathbf{c} = \mathbf{0}$ are 1-cocycles in $Z^1(\Gamma)$



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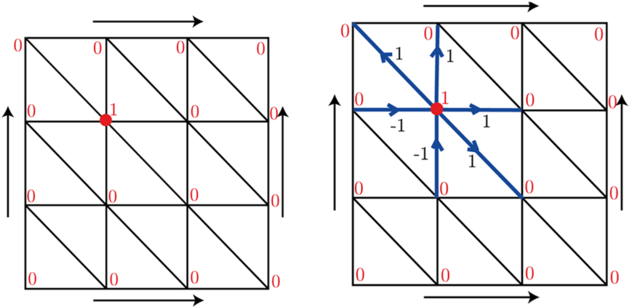
$$< \delta^k \mathbf{c}, w >=< \mathbf{c}, \partial_{k+1} w >$$

δ^k gives rise to a classification of 1-cochains:

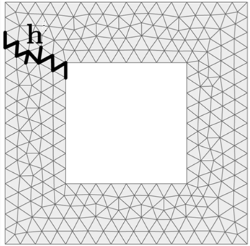
2. 1-cochains **that are coboundary** of a 0-cochain are 1-coboundaries in $B^1(\Gamma)$



Cohomology group

All k-coboundaries are k-cocycles, but the converse is not true (as not all irrotational fields are gradients when the domain is not simply connected)

Example:

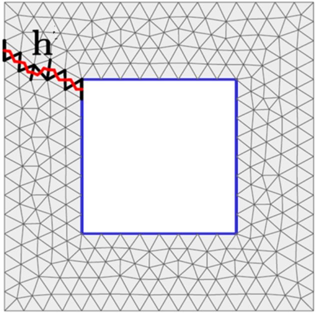


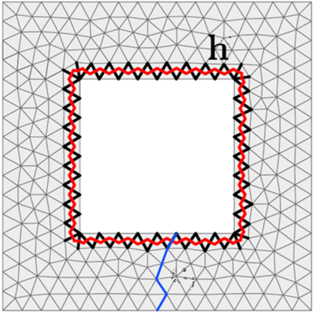
The first cohomology group is defined as the following **quotient group**

$$H^1(\Gamma) := Z^1(\Gamma)/B^1(\Gamma)$$

Relative cohomology takes into account **boundary conditions**: the current $\boldsymbol{J}$ cannot flow outside Γ

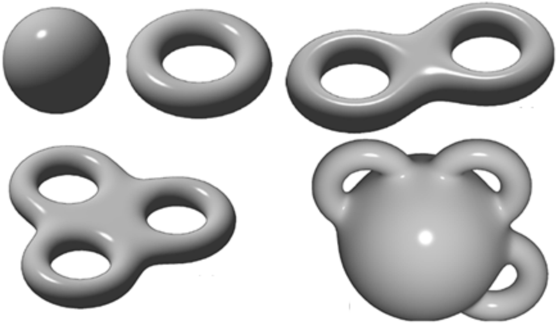
Example: $\mathbf{h} \in H^1(\Gamma, \partial\Gamma)$





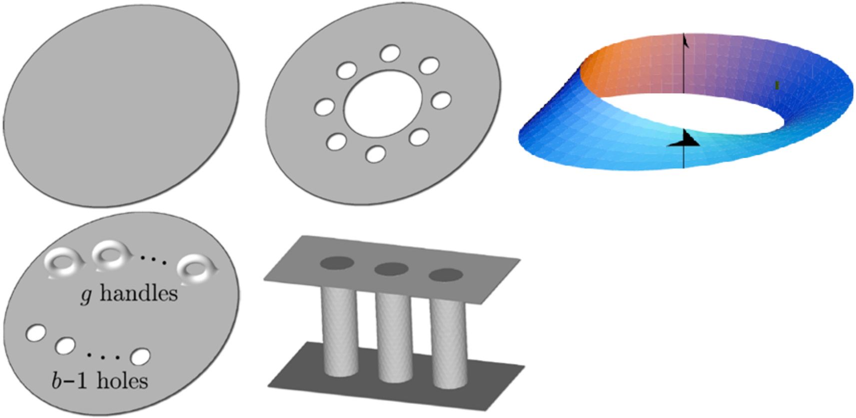
How to compute relative generators?

For closed surfaces they are absolute generators !



How to compute relative generators?

Harder ! **No combinatorial algorithm** has been devised to compute relative cohomology generators



Review of past work: Hiptmair-Ostrowski algorithm

Hiptmair-Ostrowski algorithm

- **Input**: closed and connected triangulated surface Γ
- **Output**: generators of $H_1(\Gamma)$

Introduced in:

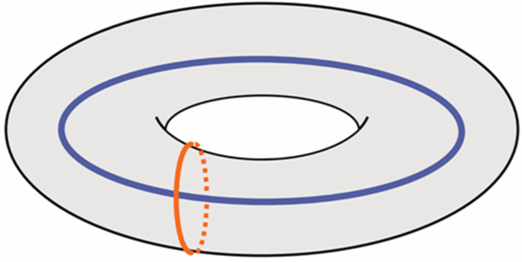
- *R. Hiptmair, J. Ostrowski, Generators of $H_1(\Gamma_h, \mathbb{Z})$ for Triangulated Surfaces: Construction and Classification, SIAM J. Comput., Vol. 31, No. 5, pp. 1405-1423 (2002)*

Reinvented several times, for example:s

- *D. Eppstein, Dynamic generators of topologically embedded graphs. Proc. ACM-SIAM Symp. on Discrete Algorithms, pp. 599–608 (2003)*
- *G. Rubinacci, A. Tamburrino, Automatic Treatment of Multiply Connected Regions in Integral Formulations, IEEE Trans. Magn., vol. 46, no. 8, pp. 2791-2794 (2010)*

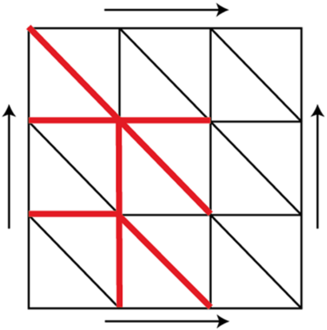
Hiptmair-Ostrowski algorithm

Example: find generators for $H_1(\Gamma)$, where Γ is a **torus**



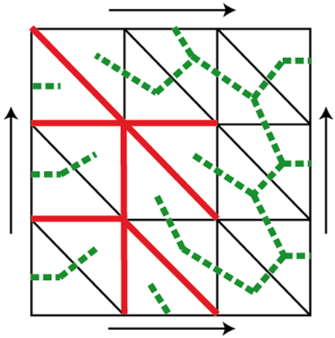
Hiptmair-Ostrowski algorithm

1. Produce a **spanning tree** T of 1-skeleton of Γ (usually by Breadth First Search)



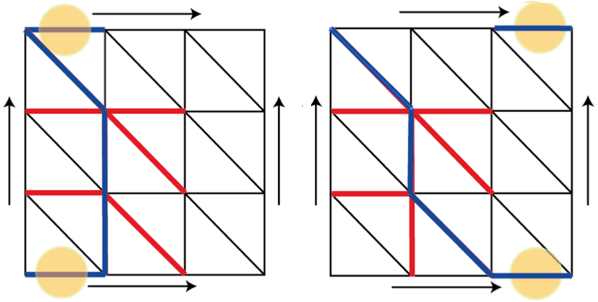
Hiptmair-Ostrowski algorithm

2. Produce a spanning tree $\tilde{T}$ of the **dual** 1-skeleton



Hiptmair-Ostrowski algorithm

3. Add each edge e neither in T nor in $\tilde{T}$ to the tree



The (unique) cycle in T is the support of the homology generator

Hiptmair-Ostrowski algorithm for cohomology?

Usually in numerical analysis and engineering applications the **cohomology generators** are needed in place of homology generators

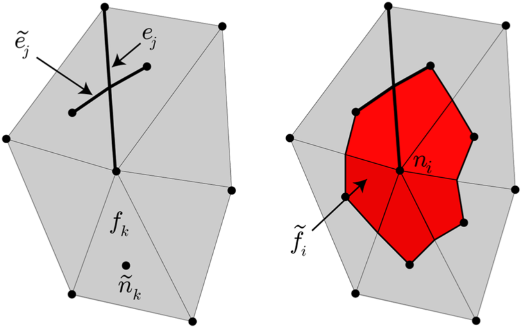
Natural question: is there an algorithm such that

- **Input**: triangulated surface with boundary Γ and
- **Output**: representatives of $H^1(\Gamma, \partial\Gamma^c)$

Hiptmair-Ostrowski algorithm for cohomology

If Γ is closed, leverage **Poincaré-Lefscthez duality**

$$H^1(\Gamma) \cong H_1(\tilde{\Gamma})$$

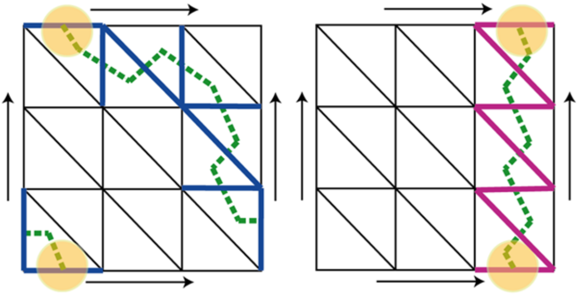


Hiptmair-Ostrowski algorithm for cohomology

If Γ is closed, leverage **Poincaré-Lefscthez duality**

$$H^1(\Gamma) \cong H_1(\tilde{\Gamma})$$

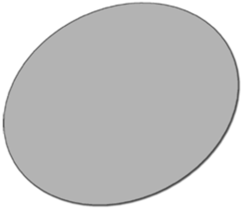
Example: algorithm on the torus, spanning trees T and $\tilde{T}$ as before

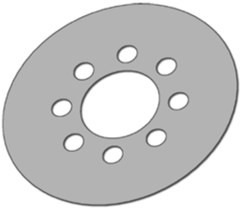


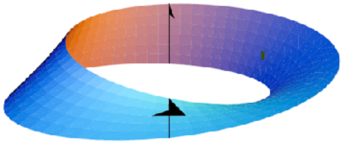
Find **dual cycle** on $\tilde{T}$ and use $H^1(\Gamma) \cong H_1(\tilde{\Gamma})$

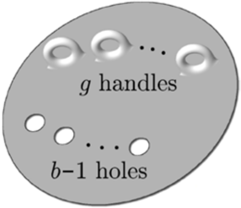
Hiptmair-Ostrowski algorithm for open surfaces

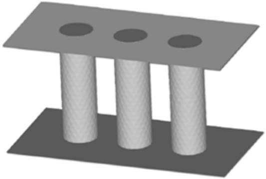
Harder ! **No combinatorial algorithm** has been devised for this case











Main algorithm overview

Main algorithm overview

Goal: Compute generators for $H^1(\Gamma, \partial\Gamma^{\mathrm{c}})$

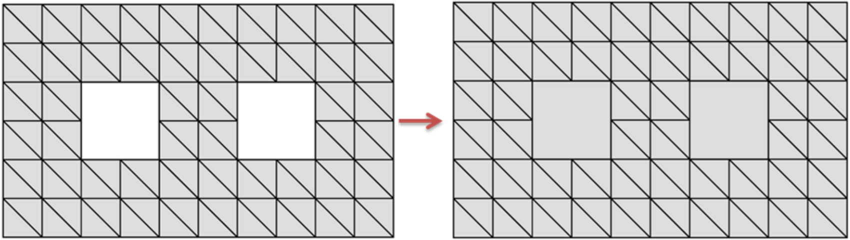
A **three-step** approach:

1. Compute N^{ha} generators $\mathbf{h}_j^{\mathrm{ha}}$ due to "handles"
2. Compute N^{h} generators $\mathbf{h}_k^{\mathrm{h}}$ due to "holes"
3. Compute N^{c} generators $\mathbf{h}_l^{\mathrm{c}}$ due to "contacts"

Compute generators due to "handles"

Less easy task: proceed in two further steps

1. "Glue" **topological disks** (i.e. polygons) along the connected components of $\partial\Gamma$ to obtain a closed surface $\bar{\Gamma}$



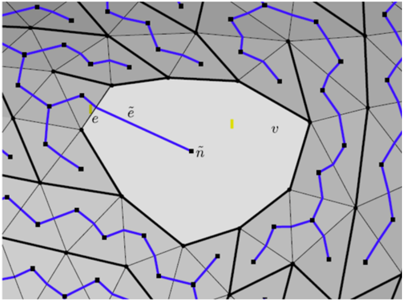
Γ $\qquad\qquad$ $\bar{\Gamma}$

Compute generators due to "handles"

Less easy task: proceed in two further steps

2. Apply the same ideas of the standard HO-algorithm on $\bar{\Gamma}$

   Warning: assure that generators do not pass through the attached disks



Characterization of cohomology generators

Universal Coefficient Theorem for cohomology

If a chain complex C of free Abelian groups has homology groups $H_n(C;G)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C;G)$ are determined by the exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(C);G) \to H^n(C;G) \xrightarrow{\varphi} \mathrm{Hom}(H_n(C);G) \to 0.$$

with $\varphi(\mathbf{h}) \mapsto < \mathbf{h}, \cdot >$

Strategy: find a set $\{\mathbf{h}_1^{\mathrm{ha}}, \ldots, \mathbf{h}_{N^{\mathrm{ha}}}^{\mathrm{ha}}\}$ such that

$$\langle \mathbf{h}_i^{\mathrm{ha}}, c_j \rangle = \delta_{i,j}$$

with $\{c_j\}_{j=1}^{N^{\mathrm{ha}}}$ a basis of (**the free part**) of $H_1(\bar{\Gamma};\mathbb{Z})$

Challenge: torsion subgroup

Structure of finitely generated abelian groups

Let G be a finitely generated abelian group. There exists some natural number $m \geq 0$, and some positive natural numbers $n_1, \ldots, n_q$, such that G is isomorphic to the direct sum

$$\mathbb{Z}^m \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_q\mathbb{Z},$$

where n_i divides n_{i+1} for all $1 \leq i \leq q-1$.

Corollary: the homology group $H_1(\bar{\Gamma}; \mathbb{Z})$ is isomorphic to a **direct sum**

$$H_1(\bar{\Gamma}; \mathbb{Z}) \cong \mathbb{Z}^m \oplus T$$

where T is the torsion subgroup of $H_1(\bar{\Gamma}; \mathbb{Z})$ and $\mathbb{Z}^m$ is the free part (**a free abelian group**) of dimension $\beta_1(\bar{\Gamma})$

A glimpse of Discrete Morse Theory

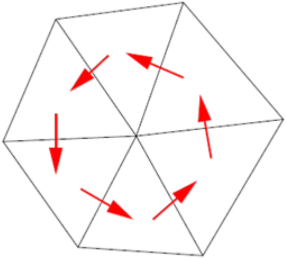
Acyclic matching

A 1-matching M on $\bar{\Gamma}$ is a set of **matched** *pairs (f, e) such that:*

1. *If $(f, e) \in M$, then $e \prec f$.*
2. *If $(f_1, e), (f_2, e) \in M$, then $f_1 = f_2$.*

A matching M is **acyclic** *if there does not exist a cycle of of the form*

$$f_1 \succ e_1 \prec f_2 \succ e_2 \prec \cdots \prec f_l \succ e_l \prec f_1$$



Acyclic matching from spanning trees

Lemma

Let T be a spanning tree of $\bar{\Gamma}$.
There exist an acyclic matching M_T such that every vertex of T, except for one arbitrary vertex, is matched in M_T.

Morse boundary formula

Theorem

Let M be an acyclic matching on $\bar{\Gamma}$, and denote by d_m the number of **non-matched** m-cells of $\bar{\Gamma}$. For every critical 2-cell f and 1-cell e, let $S(f,e)$ be the set of all sequences s of matched cells of the form

$$s := (f \succ e_1 \prec f_1 \succ e_2 \prec f_2 \succ \cdots \succ e_l \prec f_l \succ e).$$

Then, $\bar{\Gamma}$ is homotopy equivalent to a CW complex Σ with exactly d_m cells of dimension m, and the corresponding **Morse incidence number** $\iota^M(f,e)$ in Σ is given by

$$\iota^M(f,e) := \sum_{s \in S(f,e)} < \partial_2 w(s), e >,$$

where $w(s)$ is the **weight** 2-chain computed from each sequence s.

Applying the Morse framework

Σ consists of:

- One 0-cell, the unique non-matched vertex of $\bar{\Gamma}$ in M_T.
- $(|E| - |V| + 1) - (|F| - 1) = 2 - \chi$ 1-cells, where χ is the **Euler characteristic** of $\bar{\Gamma}$.
- One 2-cell, the unique non-matched face (a vertex, by duality) of $\bar{\Gamma}$ in $M_{\tilde{T}}$.

Observation: Σ is homotopy equivalent to $\bar{\Gamma}$ and the boundary of $\bar{\Gamma}$ is **empty** $\Rightarrow$ each Morse incidence number $\iota^M(f, e)$ in Σ **can only take one of two possible values**:

1. $\iota^M(f, e) = 0$.
2. $\iota^M(f, e) = 2\eta_e$, with $\eta_e \in \{-1, +1\}$.

Homology characterization of Σ (and hence of $\bar{\Gamma}$)

Lemma

Let E_c^{II} be the set of critical 1-cells such that $\iota^M(f,e) \neq 0$ in Σ. Denote by $\mathbf{g} := 2 - \chi$ the **genus** of Σ. We distinguish between the following two cases:

1. If $|E_c^{\mathrm{II}}| = 0$, then $H_1(\Sigma;\mathbb{Z}) = \mathbb{Z}^g$.
2. If $|E_c^{\mathrm{II}}| > 0$, then $H_1(\Sigma;\mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$. In this case, the **torsion generator** c^* is given by

$$c^* := \sum_{e \in E_c^{\mathrm{II}}} \eta_e \, e,$$

where each $\eta_e \in \{-1,+1\}$ is the sign of the Morse coefficient $\iota^M(f,e)$

Missing piece: identify the 1-cells of Σ in E_c^{II}

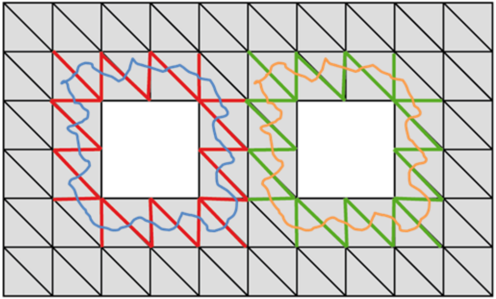
Easy ! Simply algorithm:

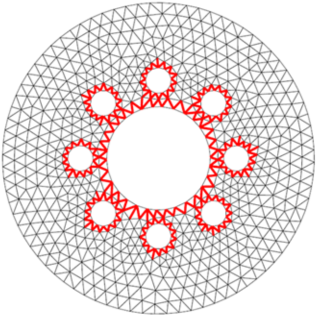
1. pick any triangle, assign any orientation to it
2. then go to neighbouring triangles (via some algorithm, it does not matter), and decide the orientation for those, which is forced to be one or another due to already oriented neighbors
3. if at some point you find inconsistency, that is, several neighbors force a single triangle to have different orientations, then the surface is non-orientable

Otherwise, when there are no more neighbors, you have found an orientable connected component of the mesh !

Compute generators due to "holes"

Very easy ! Simply select the edges that go outside all connected components except one





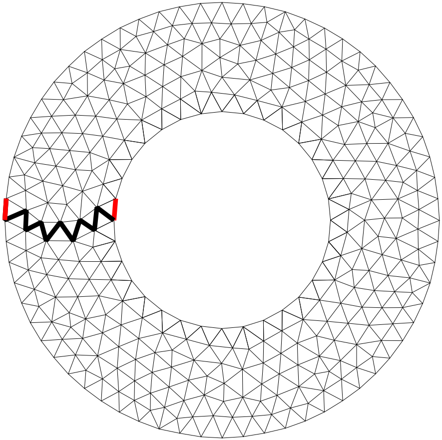
First decomposition Lemma

$$H^1(\Gamma, \partial\Gamma) \cong H^1(\bar{\Gamma}) \oplus \left\langle [\mathbf{h}_1^{\mathrm{h}}], \ldots, [\mathbf{h}_{N^{\mathrm{h}}}^{\mathrm{h}}] \right\rangle$$

Compute generators due to "contacts"

Also easy, modulo a technical detail

1. If $|E_c^{\mathrm{II}}| = 0$, then simply select one generator for all contact regions except one (called **ground contact**)



Compute generators due to "contacts"

This is also easy, modulo a technical detail involving the presence of a **torsion generator**

1. If $|E_c^{\mathrm{II}}| = 0$, then simply select one generator for all contact regions except one (called **ground contact**)

2. If $|E_c^{\mathrm{II}}| > 0$, there an additional generator associated with the chosen ground contact and the **torsion generator**

Second decomposition Lemma

$$H^1(\Gamma, \partial\Gamma^c) \cong H^1(\Gamma, \partial\Gamma) \oplus \langle [\mathbf{h}_1^{\mathrm{c}}], \ldots, [\mathbf{h}_{N^{\mathrm{c}}}^{\mathrm{h}}] \rangle$$

Summing up

We have the following isomorphism

Cohomology characterization

$$
\begin{aligned}
H^1(\Gamma, \Gamma^c) \cong\ & \langle [\mathbf{h}_1^{\mathrm{ha}}], \ldots, [\mathbf{h}_{N^{\mathrm{ha}}}^{\mathrm{ha}}] \rangle && \text{(generators due to ``handles'')} \\
\oplus\ & \langle [\mathbf{h}_1^{\mathrm{h}}], \ldots, [\mathbf{h}_{N^{\mathrm{h}}}^{\mathrm{h}}] \rangle && \text{(generators due to ``holes'')} \\
\oplus\ & \langle [\mathbf{h}_1^{\mathrm{c}}], \ldots, [\mathbf{h}_{N^{\mathrm{c}}}^{\mathrm{h}}] \rangle && \text{(generators due to ``contacts'')}
\end{aligned}
$$

Conclusion and implications

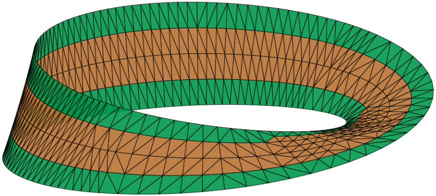
Summary

- Novel algorithm to compute $H^1(\Gamma, \partial\Gamma^c)$ with linear complexity
- Applications to EFIE and topological Data Analysis
- Handles non-orientable surfaces and torsion

  Blunder: In

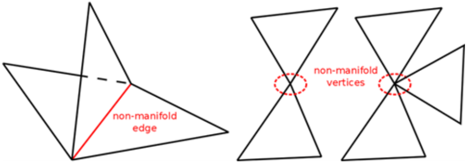
  B. Hofmann, T. F. Eibert, F. P. Andriulli, S. B. Adrian, A Low-Frequency Stable, Excitation Agnostic Discretization of the Right-Hand Side for the Electric Field Integral Equation on Multiply-Connected Geometries, IEEE Transactions on Antennas and Propagation, Vol. 71, No. 2, (2023)

  they say that **green triangles** are the support of a global loop

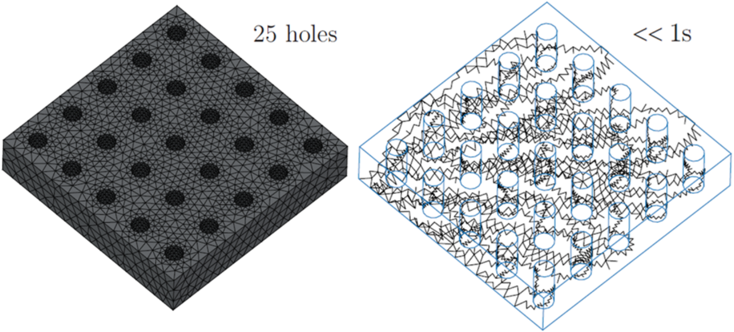


Glimpses on future work

- Extension to non-manifold surfaces



- Minimal cohomology basis generation



References

S. Pitassi, *Generators of $H^1(\Gamma, \partial\Gamma^c)$ with $\partial\Gamma^c \subset \partial\Gamma$ for Triangulated Surfaces Γ: Construction and Classification of Global Loops*, arXiv preprint (2025).